MTH 201: Multivariable Calculus and Differential Equations

Problem Set 2: Surface Integrals

1. Find parametrizations for the following surfaces.

   (a) The cap cut from the sphere $x^2 + y^2 + z^2 = 9$ by the cone $z = \sqrt{x^2 + y^2}$.

   (b) The surface cut from the parabolic cylinder $y = x^2$ by the planes $z = 0$, $z = 3$, and $y = 2$.

   (c) The portion of the cone $z = z\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$.

   (d) The portion of the plane $x - y + 2z = 2$ inside the cylinder $x^2 + z^2 = 3$.

   (e) The portion of the cylinder $y^2 + (z - 5)^2 = 25$ between the planes $x = 0$ and $x = 10$.

   (f) The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

   (g) The hyperboloid of two sheets $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + 1$.

2. Use a parametrization to the express area of the surface as a double integral and then evaluate the integral.

   (a) The cap cut from the paraboloid $z = 2 - x^2 - y^2$ by the cone $z = \sqrt{x^2 + y^2}$.

   (b) The lower position cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.

   (c) The portion of the plane $z = -x$ inside the cylinder $x^2 + y^2 = 4$.

   (d) The portion of the cone $z = \sqrt{x^2 + y^2}/3$ between $z = 1$ and $z = 4/3$.

3. Compute the surface area of the following surfaces.

   (a) The ellipse cut from the plane $z = cx$ by the cylinder $x^2 + y^2 = 1$.

   (b) The portion of $x^2 = 2z$ that lies above the traingle bounded by $x = \sqrt{3}$, $y = 0$, and $y = x$ in the xy-plane.

   (c) The cap cut from the sphere $x^2 + y^2 + z^2 = 2$ and $z = \sqrt{x^2 + y^2}$.

   (d) The surface in the first octant cut from the cylinder $y = (2/3)z^{3/2}$ by the planes $x = 1$ and $y = 16/3$.

   (e) The portion of the paraboloid $x = 4 - y^2 - z^2$ that lies above the ring $1 \leq y^2 + z^2 \leq 4$ in the yz-plane.

   (f) The portion of the cone $z = \sqrt{x^2 + y^2}$ that lies over the region between the circle $x^2 + y^2 = 1$ and the ellipse $9x^2 + 4y^2 = 36$ in the xy-plane.

4. Integrate the given function over the given surface.

   (a) $H(x, y, z) = yz$ over the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$.

   (b) $G(x, y, z) = xyz$ over the rectangular solid bounded by the planes $x = \pm a$, $y = \pm b$, and $z = \pm c$.

   (c) $G(x, y, z) = x^2$ over the unit sphere.

   (d) $G(x, y, z) = x$ over the surface given by $z = x^+y$, for $(x, y) \in [0, 1] \times [-1, 1]$. $H(x, y, z) = x^2\sqrt{5 - 4z}$ over the the parabolic dome $z = 1 - x^2 - y^2$.

   (e) $G(x, y, z) = x + y + z$ over the portion of the plane $2x + 2y + z = 2$ in the first octant.

   (f) $G(x, y, z) = z$ over the cylindrical surface $y^2 + z^2 = 4$, $z \geq 0$, $x \in [1, 4]$.

5. Find the flux across the surface in the given direction. Use a suitable parametrization of the surface, wherever required.

   (i) $F = xi + yj + zk$ outward through the portion of the cylinder $x^2 + y^2 = 1$ cut by the planes $z = 0$ and $z = a$.

   (ii) $F = 4xi + 4yj + 2k$ outward through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.

   (iii) $F = xzi + yzj + k$ across the surface of the upper cap cut from the solid sphere $x^2 + y^2 + z^2 \leq 25$ by the plane $z = 3$.

   (iv) $F = -xi - yj + z^2k$ outward (normal away from the z-axis) through the portion of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.

   (v) $F = x^2j - xzk$ outward (normal away from the yz-plane) through the surface cut from the parabolic cylinder $y = x^2$, $x \in [-1, 1]$, by the planes $z = 0$ and $z = 2$.

   (vi) $F = 2xyi + 2yzj + 2xzk$ upward across thr portion of the plane $x + y + z = 2a$ that lies above the square $[0, a] \times [0, a]$ in the xy-plane.